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## ASYMPTOTIC METHOD IN TWO-DIMENSIONAL PROBLEMS OF ELECTROELASTICITY

**Purpose.** Generalization of the asymptotic method for solving two-dimensional problems of electroelasticity. Accounting for electric charges arising from deformation on the surfaces of piezoelectric materials. Checking the possibility of taking into account the magnetic field and the opposite effect when exposed to an electric field.

**Methodology.** The mathematical model of the piezoelectric material is described using the equilibrium equations, the electro-elastic state, and the Cauchy relations. A small parameter is introduced as a ratio of the physical characteristics of the material. Transformations of coordinates and desired functions depending on the specified parameter are proposed.

**Findings.** The introduction of these transformations allowed splitting the initial boundary-value problem into two components with different properties. Each of them contains both mechanical and electrical components. The solution is sought as a superposition of solutions of both types. Each of the types of stress-strain states contains the main function and an auxiliary one. The expansion of the desired functions in rows by parameter  $\varepsilon$  and the construction of asymptotic sequences lead to the fact that in each approximation the main functions are sought from the Laplace or Poisson equations. Auxiliary ones are found by integration. The analysis of the boundary conditions is given. It is shown that they can almost always be formulated for basic functions.

**Originality.** The method proposed earlier by the authors for reducing the boundary value problems of linear and nonlinear elasticity theory to the sequential solution of potential theory problems is generalized for the case of modern piezoelectric materials described by electroelasticity equations.

**Practical value.** With the help of the proposed approach, analytical solutions of practically important problems of electroelasticity can be obtained; estimates of the stress-strain state of products from piezoelectric materials are carried out. The results can be used as null approximations in numerical calculations.

**Keywords:** *electroelasticity, interaction, analytical solution, piezoelectric, asymptotic method, two-dimensional problems, piezoelectromagnetic elements*

**Introduction.** Active materials, first of all piezoelectric and piezo-electro-magnetic ones, are often used as functional parts of different electronic devices including sensors, transducers and actuators. This is due to the fact that such materials are able to change their shape under the action of an electric or magnetic field. In many cases, the dimensions of the devices mentioned are extremely small, but nevertheless they can be exposed to very large mechanical, electric and magnetic fields.

Moreover, these devices are usually constructed of elements which can be manufactured of different materials (piezoelectric or piezoelectromagnetic elements, electrodes, and others).

In practice, the results of solving model problems taking into account the use of active materials show that the stress-strain state usually depends on the geometric characteristics, electrical or magnetic load, as well as on the material properties [1].

**Literature review.** The use of piezoelectrics in modern technology forces researchers to consider electro-electromagnetic elasticity as a single science combining electrodynamics, the theory of magnetism and continuum mechanics, which are practically studied separately. This is due to the fact that during the deformation of some materials, electric charges pro-

portional to the deformation appear on their surfaces. But it turns out that one cannot neglect the opposite effect, which is manifested by the appearance of mechanical stresses under the influence of an electric field.

This problem was considered by Parton V.V. in the work "Electromagnetic resilience of piezoelectrically conductive bodies". His works outline the main provisions on the theme under study.

The issue of using piezoelectrics in designing modern machinery is highly relevant. In this regard, quite a lot of authors pay attention to solving various kinds of model problems in the described theme.

In [2], a multilayer piezomagnetic/piezoelectric composite with periodic interphase cracks subjected to magnetic or electric fields in a plane was studied.

In [3], an interphase crack in the case of an infinite piezoelectric bi-material is considered. Initially it is believed that the crack is filled with an electrically conductive fluid. The authors take into account the contact of the crack faces, as well as the electric field in the contact zone. Often when considering such a problem (within the framework of an open model), a peculiarity arises that leads to the case of physically unrealistic mutual penetration of materials. The authors in their publication proposed for consideration a contact model for a crack between two isotropic materials, which eliminates this drawback.

In [4], the authors investigated the effect of surface piezoelectricity on out-of-plane (anti-plane) deformations of a hexagonal piezoelectric material weakened by a crack.

In the article [5], the authors investigate electroelastic features at the apex of a rectilinearly polarized piezoelectric wedge. The methods of the three-dimensional theory of piezoelectricity are used.

An analytical solution of magnetothermoelastic problems of a piezoelectric hollow cylinder placed in an axial magnetic field is presented in [6]. The cylinder is subjected to arbitrary thermal shock, mechanical stress and transient electrical excitation.

In [7], the problem of an electric and stressed state in an orthotropic electroelastic space with a circular crack with uniform force and electric loads was considered. The solution of the problem is obtained by using the triple Fourier transform and the Fourier transform of the Green function for an infinite piezoelectric medium.

**Unsolved aspects of the problem.** The works mentioned above demonstrate the widespread use of piezoelectrics and, as a result, the attention of researchers to their behavior [8, 9]. Of interest are various approaches to solving problems that take into account the complex properties of materials.

Since isotropic materials in electroelasticity are not of interest, naturally, special attention should be paid to anisotropic materials and materials with strong anisotropic activity. In this case, as a small parameter in the asymptotic analysis, it was proposed to consider the ratio of the rigidity characteristics of the material.

To take into account possible relationships between the components of the displacement vector and the rate of their change along the coordinates, it is proposed to introduce affine transformations depending on the specified parameter. The form of these transformations shows that the solutions of the corresponding systems of equations obtained for asymptotic integration have different properties. This is manifested in the difference in the order of the components, as well as in the variability of the solutions.

Each of their limiting systems of equations has a lower order than the original system. These limiting systems, as well as the corresponding asymptotic processes, are considered as complementary. Solutions of boundary value problems are sought as a superposition of the components corresponding to these types of stress-strain state.

In this article, a new asymptotic method is proposed for solving problems of electroelasticity.

**Formulation of the problem.** Let two mutually perpendicular planes of elastic symmetry pass through each point of a uniform anisotropic plate. Assuming that these planes are perpendicular to the Cartesian coordinate axes  $x, y$ , respectively, we obtain the following equilibrium equations, electrostatics, electroelastic state, and the Cauchy relations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0; \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \quad (1)$$

$$\begin{aligned} \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} &= 0; \\ \frac{\partial \dot{Y}_x}{\partial y} - \frac{\partial \dot{Y}_y}{\partial x} &= 0; \end{aligned} \quad (2)$$

$$\begin{aligned} e_x &= s_{11}^D \sigma_x + s_{12}^D \sigma_y + g_{11}^{\sigma_0} D_x; \\ e_y &= s_{21}^D \sigma_x + s_{22}^D \sigma_y + g_{12}^{\sigma_0} D_x; \\ \gamma_{xy} &= s_{66}^D \tau_{xy} + g_{26}^{\sigma_0} D_y; \\ \dot{Y}_x &= -g_{11}^{\sigma_0} \sigma_x - g_{12}^{\sigma_0} \sigma_y + \beta_{11}^{\sigma_0} D_x; \\ \dot{Y}_y &= -g_{26}^{\sigma_0} \tau_{xy} + \beta_{22}^{\sigma_0} D_y; \\ e_x &= \frac{\partial U}{\partial x}; \quad e_x = \frac{\partial V}{\partial y}; \end{aligned} \quad (3)$$

$$\gamma_{xy} = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}. \quad (4)$$

Here  $\sigma_x, \sigma_y, \tau_{xy}$  are normal (tangential) stresses;  $D_x, D_y$  and  $\dot{Y}_x, \dot{Y}_y$  are the components of the induction vector and the electric field strength;  $s_{ij}$  stands for deformation coefficients of the material of the body, measured at a constant induction of the electric field;  $g_{kj}^{\sigma_0}$  is piezoelectric deformation and tension modules, measured at constant induction voltages;  $\beta_{kl}^{\sigma_0}$  is dielectric susceptibility coefficients measured at constant voltages;  $U, V$  are the components of the plate displacement vector. From the first equation of system (2) it follows that there exists some scalar function  $\varphi = \varphi(x, y)$ , such that

$$D_1 = D_x = \frac{\partial \varphi}{\partial y}; \quad D_2 = D_y = -\frac{\partial \varphi}{\partial x}.$$

**Results.** The solution of a particular boundary value problem can be reduced to the integration of the system of equations

$$\begin{aligned} U_{xx} + \varepsilon u_{yy} + \varepsilon m V_{xy} - (a_{11} - \varepsilon a_{26}) \varphi_{xy} &= 0; \\ \varepsilon V_{xx} + q V_{yy} + \varepsilon m U_{xy} + \varepsilon a_{26} \varphi_{xx} - q a_{12} \varphi_{yy} &= 0; \\ -(a_{11} - \varepsilon a_{26}) U_{xy} + \varepsilon a_{26} V_{xx} - q a_{12} V_{yy} + \varepsilon b_{22} \varphi_{xx} + b_{11} \varphi_{yy} &= 0; \end{aligned} \quad (5)$$

$$\varepsilon = \frac{G}{B_1}; \quad q = \frac{B_2}{B_1}; \quad m = 1 + \frac{\nu_2 B_1}{G} = 1 + \frac{\nu_1 B_2}{G};$$

$$a_{11} = g_{11}^{\sigma_0 D} + \nu_2 g_{12}^{\sigma_0 D};$$

$$a_{26} = g_{26}^{\sigma_0 D}; \quad a_{12} = g_{12}^{\sigma_0 D} + \nu_1 g_{11}^{\sigma_0 D};$$

$$b_{22} = a_{26}^2 + \beta_{22}^{\sigma_0} \frac{\sigma}{G};$$

$$b_{11} = g_{11}^{\sigma_0 D} a_{11} + g_{12}^{\sigma_0 D} a_{12} \frac{B_2}{B_1} + \beta_{11}^{\sigma_0} \frac{\sigma}{B_1},$$

under appropriate boundary conditions.

The components of the stress tensor and the intensity vector in this case are written as follows

$$\sigma_1 = B_1(u_x + \nu_2 V_y - a_{11} \varphi_y);$$

$$\sigma_2 = B_2(\nu_1 u_x + V_y - a_{12} \varphi_y);$$

$$\tau = G(u_y + V_x + a_{26} \varphi_x);$$

$$\tilde{E}_1 = -B_1 a_{11} u_x - B_2 a_{12} V_y + B_1 b_{11} \varphi_y; \quad (6)$$

$$\tilde{E}_2 = -G a_{26} u_y - G a_{26} V_x + G b_{22} \varphi_x.$$

Here  $\sigma_1 = \delta \sigma_x; \sigma_2 = \delta \sigma_y; \tau = \delta \tau_{xy}; \dot{Y}_1 = \delta \dot{Y}_x; \dot{Y}_2 = \delta \dot{Y}_y; B_1 = \frac{E_1 \delta}{1 - \nu_1 \nu_2}; B_2 = \frac{E_2 \delta}{1 - \nu_1 \nu_2}; G = G_x \delta; \delta$  is plate thickness. The indices  $x, y$  in equations (5) and relations (6) denote the differentiation by coordinates;  $E_1, E_2$  are modulus of elasticity along the main directions  $x, y$ ;  $G_x$  is shear modulus;  $\nu_1, \nu_2$  are Poisson's coefficients. In real orthotropic materials, the value  $\varepsilon = \frac{G}{B_1}$  is always much less than one. The attitude  $q = \frac{B_2}{B_1}$  may be different ( $q \leq 1$  or  $q \geq 1$ ), but always remains more than  $\varepsilon$ . Therefore, the quantity  $\varepsilon$  can be considered as a small parameter in the asymptotic integration of system (5). The value  $q$  in the future we will assume approximately equal to one. We introduce the following transformations

$$\begin{aligned} \xi_1 &= \alpha \varepsilon^{\frac{1}{2}} x; \quad \eta_1 = y; \quad u = U^{(1)}; \\ V &= \varepsilon^{\frac{3}{2}} V^{(1)}; \quad \varphi = \varepsilon^{\frac{3}{2}} \varphi^{(1)}; \end{aligned} \quad (7)$$

$$\xi_2 = x; \quad \eta_2 = \beta \varepsilon^{\frac{1}{2}} y; \quad u = \varepsilon^{\frac{3}{2}} U^{(2)}; \\ V = V^{(2)}; \quad \varphi = \varepsilon^2 \varphi^{(2)}. \quad (8)$$

Transformations (7, 8) show that solutions of the system of equations obtained from (5) after the introduction of transformations (7) (or (8)) change relatively slowly along the coordinate  $x$  (or  $y$ ) in comparison with similar solutions of the system obtained after applying other transformations. In the stress state of the first type (slowly varying along the coordinate  $x$ ; transformations (7)) the main role is played by the displacement component  $u$ , the normal stress  $\sigma_1$ , the component of the tangential stress  $\tau$ , depending on the displacement  $u$ . In the stressed state of the second type (transformation (8)), the displacement  $V$ , stress  $\sigma_2$ , and the component of the shear stress  $\tau$ , depending on the displacement component  $V$ . The total shear stress consists of the sum of both components; it is the link between these two types of stress states that is realized through it. Depending on the loading, one of them has the character of a boundary layer. Thus, during mechanical loading of piezomaterials, when the boundary conditions are specified in stresses, displacements or their combinations, the solutions of the corresponding boundary problems will be represented as a superposition of solutions of these two types of stress-strain state

$$u = u^{(1)} + u^{(2)}; \quad V = V^{(1)} + V^{(2)}; \quad \varphi = \varphi^{(1)} + \varphi^{(2)}.$$

When looking for functions,  $u^{(n)}$ ,  $V^{(n)}$ ,  $\varphi^{(n)}$  ( $n = 1, 2$ ) in the form of a rows in powers of a parameter  $\varepsilon$ , it is necessary to choose the corresponding asymptotic sequences. The form of the asymptotic sequence is determined by the structure of equations (5) and the order of the  $\varepsilon$  residual in the boundary conditions that arise after solving the problem in the zero approximation ( $\varepsilon \rightarrow 0$ ). To take into account all possible cases, we will define these functions in the form of series by parameter  $\varepsilon^{1/2}$  (it is clear from transformations (7, 8) that series by lower degrees of the parameter cannot occur)

$$U^{(n)} = \sum_{j=0}^{\infty} \varepsilon^{j/2} U^{n,j}; \\ V^{(n)} = \sum_{j=0}^{\infty} \varepsilon^{j/2} V^{n,j}; \quad (9) \\ \varphi^{(n)} = \sum_{j=0}^{\infty} \varphi^{n,j}, \quad (n=1,2).$$

The coefficients  $\alpha$ ,  $\beta$  are also represented in the form of rows with respect to the parameter  $\varepsilon^{1/2}$ , and  $\alpha_0 = \beta_0 = 1$ , and the coefficients  $\alpha_j$ ,  $\beta_j$  ( $j = 1, 2, \dots$ ) are found from the same conditions as for elastic materials [10], namely: in each of the approximations on the left side of the equations to determine the basic functions  $U^{l,j}$ ,  $V^{2,j}$  the Laplace operators of these functions should remain, and in the first part there are no components of the displacement vector or their derivatives (in the elastic problem for determining the main functions in each of the approximations, the right-hand sides zero). The auxiliary functions  $V^{1,j}$ ,  $U^{2,j}$ ,  $\varphi^{1,j}$ ,  $\varphi^{2,j}$  through the main are expressed by simple integration. In particular, when  $q = 1$  the coefficients  $\alpha_j = \beta_j$  ( $j = 1, 2, \dots$ ) are determined by the formula

$$\alpha_j = \frac{1}{2} m^2 \left( \omega_{j-4} + \sum_{k=0}^{j-4} \alpha_k \omega_{j-4-k} \right); \\ \omega_s = \alpha_s + (1-m) \sum_{n=0}^{s-2} (2\alpha_n - c_n) \omega_{s-n-2}; \quad (10) \\ c_n = \sum_{s=0}^n \alpha_s \alpha_{n-s}; \quad (s \geq 1 \text{ at } s < 0, \omega_s = 0).$$

From this formula it follows that the coefficient  $\alpha$  is expanded in a series of powers  $\varepsilon$ , but not  $\varepsilon^{1/2}$ . If  $m = 1$  (in this case, the Poisson ratios are assumed to be equal to zero), then  $\alpha$  it is expanded in a series in powers  $\varepsilon^2$ . We substitute the transformations (7) into the system (5) and use the corresponding expansions taking into account (10). After splitting the resulting system by parameter  $\varepsilon^{1/2}$  we arrive at an infinite system of equations for the functions  $U^{l,j}$ ,  $V^{1,j}$ ,  $\varphi^{1,j}$  ( $j = 0, 1, \dots$ ). In this case, we assume that  $\alpha_{11} \sim \varepsilon \beta_{11}$ ,  $b_{22} \sim \varepsilon^2 b_{11}$ ,  $a_{12} \sim a_{26} \sim \varepsilon^3 b_{11}$ . We give these equations for the first three approximations ( $j = 0, 1, 2$ ).

$$U_{\xi\xi}^{1,0} + U_{\eta\eta}^{1,0} = 0; \\ qV_{\eta\eta}^{1,0} + mU_{\xi\eta}^{1,0} = 0; \\ -a_{11}U_{\xi\eta}^{1,0} + \varepsilon b_{11}\varphi_{\eta\eta}^{1,0} = 0; \quad (11) \\ U_{\xi\xi}^{1,1} + U_{\eta\eta}^{1,1} = 0; \\ qV_{\eta\eta}^{1,1} + mU_{\xi\eta}^{1,1} = 0; \\ -a_{11}U_{\xi\eta}^{1,1} + \varepsilon b_{11}\varphi_{\eta\eta}^{1,1} = 0; \\ qV_{\eta\eta}^{1,2} + mU_{\xi\eta}^{1,2} = 0; \\ -a_{11}U_{\xi\eta}^{1,2} + \varepsilon b_{11}\varphi_{\eta\eta}^{1,2} = 0.$$

Here and hereinafter, it is assumed that differentiation (indices  $\xi, \eta$ ) are made according to those coordinates,  $\xi_n, \eta_n$  ( $n = 1, 2$ ), whose indices coincide with the first superscripts of functions. After substituting the transformations (8) into the system (5) using the corresponding expansions and splitting in the parameter  $\varepsilon^{1/2}$ , we obtain an infinite system of equations for the functions  $U^{2,j}$ ,  $V^{2,j}$ ,  $\varphi^{2,j}$  ( $j = 0, 1, \dots$ ), which determine the solutions of the second type. From system (11) it follows that in the first two approximations ( $j = 0, 1$ ) the main functions  $U^{l,j}$  ( $V^{2,j}$  for the second stressed state) are determined from the Laplace equations (with  $q = 1$  or one of the variables is obviously replaced), and the auxiliary functions are expressed by simple integration through the main ones.

In the third approximation ( $j = 2$ ) and further, the stress state of the first type of function  $U^{1,j}$  is found from the Poisson equation with a known right-hand side, which contains only the function  $\varphi$ , which is found in the previous approximations. A similar situation takes place in the stressed state of the second type for the function  $V^{2,j}$ , but starting from the fourth approximation ( $j = 3$ ) and beyond. We use transformations (7, 8) and expansions in expressions for displacements, stresses and strengths (6) and present them in series

$$U = U^{1,0} + \varepsilon^{1/2} U^{1,1} + \varepsilon U^{1,2} + \dots; \\ V = V^{2,0} + \varepsilon^{1/2} V^{2,1} + \varepsilon V^{2,2} + \varepsilon^{3/2} (V^{2,3} + V^{1,0}) + \dots; \\ \sigma_1 = B_1 \varepsilon^{1/2} \left[ U_{\xi\xi}^{1,0} + \varepsilon^{1/2} (U_{\xi\xi}^{1,1} + \nu_2 U_{\eta\eta}^{2,0}) + \right. \\ \left. + \varepsilon (U_{\xi\xi}^{1,2} + U_{\xi\xi}^{2,0} + \nu_2 U_{\eta\eta}^{2,1} - a_{11} \varphi_{\eta\eta}^{1,0}) + \dots \right]; \\ \tilde{E}_1 = -B_1 a_{11} \varepsilon^{1/2} \left[ U_{\xi\xi}^{1,0} + \varepsilon^{1/2} U_{\xi\xi}^{1,1} + \varepsilon (U_{\xi\xi}^{1,2} + U_{\xi\xi}^{2,0}) + \dots \right] - \\ -B_2 a_{12} \varepsilon^{1/2} V_{\eta\eta}^{2,0} + B_1 b_{11} \varepsilon^{1/2} \left[ \varphi_{\eta\eta}^{1,0} + \varepsilon^{1/2} \varphi_{\eta\eta}^{1,1} + \varepsilon (\varphi_{\eta\eta}^{1,2} + \varphi_{\eta\eta}^{2,0}) + \dots \right]; \quad (12) \\ \dot{Y}_2 = \dot{Y}_{2,0} + \varepsilon^{1/2} \dot{Y}_{2,1} + \varepsilon \dot{Y}_{2,2} + \dots = \\ = G a_{26} \left[ (U_{\eta\eta}^{1,0} + V_{\xi\xi}^{2,0}) + \varepsilon^{1/2} (U_{\xi\xi}^{1,1} + U_{\xi\xi}^{2,1}) + \right. \\ \left. + (U_{\eta\eta}^{1,0} + V_{\xi\xi}^{2,0}) + \varepsilon^{1/2} (U_{\xi\xi}^{1,1} + U_{\xi\xi}^{2,1}) + \dots \right] - \\ - G b_{22} \varepsilon^2 \left[ (\varphi_{\xi\xi}^{1,0} + \varphi_{\xi\xi}^{2,0}) + \varepsilon^{1/2} (\varphi_{\xi\xi}^{1,1} + \varphi_{\xi\xi}^{2,1}) + \varepsilon (\varphi_{\xi\xi}^{1,2} + \varphi_{\xi\xi}^{2,2}) + \dots \right]; \\ \sigma_2 = B_2 \varepsilon^{1/2} \left[ V_{\eta\eta}^{2,0} + \varepsilon^{1/2} (V_{\eta\eta}^{2,1} + \nu_1 U_{\xi\xi}^{1,0}) + \dots \right]; \\ \tau = G \left[ U_{\xi\xi}^{1,0} + V_{\xi\xi}^{2,0} + \varepsilon^{1/2} (U_{\eta\eta}^{1,1} + V_{\xi\xi}^{2,1}) + \varepsilon (U_{\eta\eta}^{1,2} + V_{\xi\xi}^{2,2}) + \dots \right].$$

It was taken into account that Poisson's coefficients  $\nu_1, \nu_2$  are of order  $\varepsilon^{1/2}$ ,  $a_{12}$  – of order  $\varepsilon^2 b_{11}$ ,  $a_{11}$  – of order  $\varepsilon b_{11}$ . From

relations (11, 12) it can be seen that the stress-strain states of the first and second types are connected only through the boundary conditions. Since the main functions  $U^{2,j}$ ,  $V^{2,j}$  are determined from the Laplace (Poisson) equations, the efficiency of the method depends on whether it is possible to form the corresponding boundary value problems for finding these functions.

**Analysis of boundary conditions.** Let normal  $\sigma_1$  and tangential  $\tau$  stresses be known on the limiting line (for example,  $x = \text{const}$ ,  $V^{2,j}$ )

$$\sigma_1 = f_1(y); \quad \tau = f_2(y); \quad \varphi = f_3(y).$$

We assume that functions  $f_n(y)$  can be represented by series

$$f_n(y) = \sum_{j=0}^{\infty} \varepsilon^{j/2} f_{n,j}(y), \quad (n=1,2).$$

Then on the bounding line

$$x = \text{const}; \quad \sigma_{1,j} = f_{1,j}; \quad \tau_j = f_{2,j}; \quad \varphi_j = f_{3,j}.$$

Using the results of the previous paragraph, we arrive at integrating the equations of the stressed state of the first type ( $j = 0$ ) under the following boundary conditions for the main functions

$$U_{\xi}^{1,0} = (B_1 \varepsilon^{1/2})^{-1} f_{1,0}; \quad V_{\xi}^{2,0} = G^{-1} f_{2,0} - U_{\eta}^{1,0}.$$

Similarly, it is not difficult to obtain boundary conditions for functions  $U^{1,0}$ ,  $V^{2,0}$  in the case of a mixed problem. An analysis of the boundary conditions shows that for all boundary value problems, the boundary conditions in the zero approximation ( $j = 0$ ) of the stress state of the first type depend neither on the higher approximations nor on the solutions of the stress state equations of the second type. Therefore, the function  $U^{1,0}$  is independent of the rest. Then, by simple integration, functions  $\varphi^{1,0}$  are determined using  $U^{1,0}$ . After that, the boundary conditions for finding the function  $V^{2,0}$  are also completely determined from the Laplace equation. Solving this equation and determining the functions  $U^{2,0}$  and  $\varphi^{2,0}$ , we obtain the boundary conditions for finding the function  $U^{1,1}$ .

**Conclusions.** For elastic (viscoelastic) orthotropic materials, an asymptotic method was developed [10], which allowed us to reduce the study of problems of a mechanically deformable solid body to the sequential solution of boundary value problems of potential theory. In this paper, an attempt is made to generalize the above method to two-dimensional problems of electroelasticity.

The effectiveness of the method depends on whether it is possible to formulate the corresponding boundary value problems for the basic equations. It is shown that this can indeed be done and, therefore, the boundary-value problems of the theory of elasticity for flat orthotropic bodies reduce to successively solvable problems of the theory of potential. This opened up new opportunities for the study of many practically important problems that lacked both analytical and numerical solutions.

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## Асимптотичний метод у двовірних задачах електропружності

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**Мета.** Узагальнення асимптотичного методу для розв'язання двовірних задач електропружності. Урахування електричних зарядів, що виникають під час деформації на поверхнях п'єзоелектричних матеріалів. Перевірка можливості врахування магнітного поля та оберненого ефекту під впливом електричного поля.

**Методика.** Математична модель п'єзоелектричного матеріалу описана за допомогою рівнянь рівноваги, електропружного стану та співвідношень Коші. Уведено малий параметр як співвідношення фізичних характеристик матеріалу. Запропоновані перетворення координат і невідомих функцій, що залежать від вказаного параметру.

**Результати.** Уведення вказаних перетворень дозволило розщепити вихідну крайову задачу на дві складові з різноманітними властивостями. Кожна з них містить як механічну, так і електричну компоненту. Розв'язок відшукують у вигляді суперпозиції розв'язків обох типів. Кожен із типів напружено-деформованих станів містить основну функцію та допоміжну. Розклад шуканих функцій у ряди за параметром  $\varepsilon$  та побудова асимптотичних послідовностей призводить до того, що в кожному наближенні основні функції розшукуються з рівнянь Лапласа або Пуассона. Допоміжні знаходять інтегруванням. Проведено аналіз граничних умов. Показано, що вони практично завжди можуть бути сформульовані для основних функцій.



**Наукова новизна.** Запропонований раніше авторами метод зведення крайових задач лінійної та нелінійної теорії пружності до послідовного розв'язання задач теорії потенціалу узагальнений для випадку сучасних п'єзоелектричних матеріалів, що описані рівняннями рівноваги електропружності.

**Практична значимість.** За допомогою запропоновано-го підходу можуть бути отримані аналітичні розв'язки практично важливих задач електропружності, проведені оцінки напружено-деформованого стану виробів із п'єзоелектричних матеріалів. Результати можуть бути використані як нульові наближення при чисельних розрахунках.

**Ключові слова:** електропружність, взаємодія, аналітичний розв'язок, п'єзоелектричний, асимптотичний метод, двовимірні задачі, п'єзоелектромагнітні елементи

## Асимптотический метод в двумерных задачах электроупругости

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**Цель.** Обобщение асимптотического метода для решения двумерных задач электроупругости. Учет электрических зарядов, возникающих при деформировании на поверхностях пьезоэлектрических материалов. Проверка возможности учета магнитного поля и обратного эффекта при воздействии электрического поля.

**Методика.** Математическая модель пьезоэлектрического материала описана с помощью уравнений равновесия, электроупругого состояния и соотношений Коши. Введен малый параметр как отношение физических ха-

рактеристик материала. Предложены преобразования координат и искомым функций, зависящие от указанного параметра.

**Результаты.** Введение указанных преобразований позволило расщепить исходную краевую задачу на две составляющие с различными свойствами. Каждая из них содержит как механическую, так и электрическую компоненту. Решение разыскивают в виде суперпозиции решений обоих типов. Каждый из типов напряженно-деформированных состояний содержит основную функцию и вспомогательную. Разложение искомым функций в ряды по параметру  $\epsilon$  и построение асимптотических последовательностей приводит к тому, что в каждом приближении основные функции разыскиваются из уравнений Лапласа или Пуассона. Вспомогательные – находят интегрированием. Проведен анализ граничных условий. Показано, что они практически всегда могут быть сформулированы для основных функций.

**Научная новизна.** Предложенный ранее авторами метод сведения краевых задач линейной и нелинейной теории упругости к последовательному решению задач теории потенциала обобщен для случая современных пьезоэлектрических материалов, которые описаны уравнениями электроупругости.

**Практическая значимость.** С помощью предложенного подхода могут быть получены аналитические решения практически важных задач электроупругости, проведены оценки напряженно-деформированного состояния изделий из пьезоэлектрических материалов. Результаты могут быть использованы как нулевые приближения при численных расчетах.

**Ключевые слова:** электроупругость, взаимодействие, аналитическое решение, пьезоэлектрический, асимптотический метод, двумерные задачи, пьезоелектромагнитные элементы

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