Modeling issues in problems of the elasticity and viscoelasticity theory

T Kagadiy[1], **A Shporta**[1], **I Scherbina**[2, 4], **O Onopriienko**[3]

[1] National TU Dnipro Polytechnic, Department of Higher Mathematics, 49005, Yavornytsky D. avenue, 19, Dnipro, UKRAINE

[2] Dnipro State Agrarian and Economic University, Department of Higher Mathematics and Physics 49000, S. Efremova S., 25, Dnipro, UKRAINE

[3] Dnipro State Agrarian and Economic University, Department of Theoretical Mechanics 49000, S. Efremova S., 25, Dnipro, UKRAINE

[4]sherbinaiv@ukr.net

Abstract. The development of computer technology and information technology does not exclude preliminary analytical modeling in complex problems of mechanics. The approach proposed in this work allows one to obtain reasonable simplified equations that admit analytical or numerical-analytical solutions. Authors research are devoted to method elaboration for space problems of viscoelasticity. The approach for solving problems of nonlinear elasticity theory, when final deformations or physical nonlinear properties of materials are taken into consideration, is proposed. New perturbation method for solving of nonlinear equations in particular derivatives is suggested. Such approach is allowed to reduce the solution of complicated problems of linear elasticity and viscoelasticity to subsequently solved boundary problems of potential theory. New linear problems are investigated, in particular, problems on load transference from stringer to single layer or multilayer solids, on stress-strain state of fibrous composite with crack in matrix (plate and axisymmetric problems), numerous contact problems (pressing of hard stamp in an orthotropic plate with different anisotropy). Problems of nonlinear elasticity theory on stress-strain state of a plate with a circular hole under different types of loading are solved.

Introduction.

The problems of transferring loads from reinforcing elements to bodies with different properties are directly related to structural mechanics. Some issues of fracture of fiber-reinforced composites can also be considered. In this work, the formulation of the problem is complicated by taking into account the viscoelastic properties of the base material, which consists of two layers. Analytical solutions for spatial problems of this type are almost never found in the literature. Therefore, this investigation is actual and can be useful for further numerical calculations. The works of many scientists are devoted to the methods of solving plane and axisymmetric contact problems on the transfer of loads from stamps, linings and other reinforcing elements to bases with different properties [1, 2, 7].

The aim of research in recent years is to take into account the complex properties of materials, which brings the mathematical model closer to real problems. For example, in [3], a method was proposed for solving the problem of electroelasticity for multiply connected plates.

Building a mathematical model

Consider an axisymmetric contact problem on the transfer of a load from a rod of circular cross-section to a viscoelastic body, which consists of two orthotropic layers with cylindrical anisotropy attached to each other

$$0 \le z_1 \le h_1, h_1 \le z_2 \le h_2, r \ne \infty, z_1 = z, z_2 = z - h_1.$$

The rod is located perpendicular to the bounding plane, its middle line coincides with the axis Oz.

It is required to find the distribution law of contact stresses between the stringer and the body, as well as the forces in the stringer under the condition of its loading at the end points by longitudinal forces $P_0^{(1)}$ and $P_0^{(2)}$ (indices 1, 2 refer to the corresponding layers). At finite values h_1, h_2 from the equilibrium conditions $P_0^{(1)} = P_0^{(2)}$.

Since in spatial problems for bodies with inclusions, the model of a one-dimensional elastic bar together with a contact model along a line is not used, it is assumed that there is a model of a one-dimensional inclusion in combination with a contact model along a cylindrical surface for the base.

Since the base material is viscoelastic with cylindrical anisotropy, and the problem is formulated taking into account the axial symmetry of the loading, then the stress tensor and displacement vector do not depend on Q (rQz cylindrical coordinates).

In this case, the problem is divided into two independent ones: the deformation problem, in which one displacement component is missing, and the torsion problem. The first of them is considered in detail. After applying the Laplace transform in time to the relationship between strains and stresses, the problem is reduced to integrating the equilibrium equations with respect to transformants for each layer [5]

$$\frac{\partial^2 u}{\partial r^2} + \varepsilon_1 \frac{\partial^2 u}{\partial z^2} + \varepsilon_1 m \frac{\partial^2 \varpi}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} = 0$$

$$\varepsilon_1 \frac{\partial^2 \varpi}{\partial r^2} + \frac{\partial^2 \varpi}{\partial z^2} + \varepsilon_1 m \frac{\partial^2 u}{\partial r \partial z} + \varepsilon_1 \frac{1}{r}\frac{\partial \varpi}{\partial r} + \varepsilon_1 m \frac{1}{r}\frac{\partial u}{\partial z} = 0 \qquad (1)$$

$$\varepsilon_1 = \varepsilon \Phi(p) \quad \varepsilon = G/E_1$$

under the following boundary conditions

$$\sigma_{33} = E_3\left(\frac{\partial \varpi}{\partial z} + \nu_{31}\frac{\partial u}{\partial r} + \nu_{32}\frac{u}{r}\right) \sigma_{13} = G\left(\frac{\partial \varpi}{\partial r} + \frac{\partial u}{\partial z}\right), \sigma_{33} = \sigma_{13} = 0 \ (z = 0,\ z = h_1 + h_2)$$

$$\varpi = \varpi_c, u = 0 \quad (r = a)$$

When $z = h_1$ the layers transformants of displacement are equal to each other. The same condition is fulfilled for stresses σ_{33}, σ_{13}. All functions turns to zero at infinity. The displacement of the bar ϖ_c satisfies the relations

$$E_c F_c d^2 \varpi_c / dz^2 = q(z) \qquad N = E_c F_c d\varpi_c / dz = P_0 \qquad (z = 0,\ z = h_1 + h_2)$$

Here u, ϖ_c are the transformants of the components of the displacement vector of the corresponding layers, E_1, E_2, E_3 (G) are analogs of the elasticity (shear) moduli of the materials of the layers, taking into account the functions that appear when taking into account the viscoelasticity, $m = G + \nu_{13} E_1$.

The value ε_1 is considered as a "small" parameter [4] in the asymptotic integration of the system (1). It is really small, since ε it is a small value, and the function $\Phi(p)$ does not exceed unity for the difference creep kernels (p is the Laplace transform parameter).

$q(z) = -2\pi a \sigma_{13}(a, z)$ - the force of contact interaction between the bar and the layers, a is the radius of the bar. Since at $r = a$ the displacement transformant u and the derivative $\partial u / \partial z$ are equal to zero, then the stresses $\sigma_{13}(a,z)$ are completely determined only by the function $\partial\varpi/\partial z$:

$$q(z) = -2\pi aG(\partial\varpi/\partial r)\big|_{r=a}$$

To determine the effort in the stringer and the effort of contact interaction, the method proposed by the authors is used [4,5].

After the decomposition of the stress-strain state into two components, the boundary value problem is reduced to the following

for the first layer:

$$E_3^1\frac{\partial^2\varpi_1}{\partial z^2}+G^{(1)}\left(\frac{\partial^2\varpi_1}{\partial r^2}+\frac{1}{r}\frac{\partial\varpi_1}{\partial r}\right)=0 \qquad (2)$$

$$\text{for } z=0,\ \frac{\partial\varpi_1}{\partial z}=0,\quad \text{for } z=h_1\quad \frac{\partial\varpi_1}{\partial z}=\frac{f(r)}{E_3^{(1)}}\quad \text{for } r=a\quad \varpi_1=\varpi_c\quad \left(\frac{\partial\varpi_1}{\partial z}\Big|_{r=a}=\frac{\partial\varpi_c}{\partial z}\right)$$

$$E_cF_c d^2\varpi_c/dz^2 = -2\pi aG^1(\partial\varpi_1/\partial r)\big|_{r=a} \qquad (3)$$

$$\text{for } z=0,\ \frac{\partial\varpi_c}{\partial z}=\frac{P_0^{(1)}}{E_cF_c},\ \text{for } z=h_1\quad \frac{\partial\varpi_c}{\partial z}=\frac{\partial\varpi_1}{\partial z}\Big|_{r=a}=\frac{f(a)}{E_3^{(1)}}$$

For the second layer, similarly.

Note once again that the writing of equations through transformants coincides with the corresponding equations for an elastic material, since the additional functions that arise due to viscoelasticity are introduced into the coefficients (constants) in the equations. After applying to equations (2), (3) the Fourier cosine transform in the coordinate z with finite limits, we obtain

$$\frac{d^2\varpi_1^*}{dr^2}+\frac{1}{r}\frac{d\varpi_1^*}{dr}-\varpi_1^2\left(\frac{n\pi}{h_1}\right)\varpi_1^*(r,n)=-\frac{(-1)^n}{G^{(1)}}f(r)$$

$$E_cF_c\left(\frac{n\pi}{h_1}\right)^2\varpi_1^*(a,n)=\frac{E_cF_c}{E^{(1)}}f(a)(-1)^n-p_0^{(1)}+2\pi aG^{(1)}\frac{d\varpi_1^*}{dr}\Big|_{r=a} \qquad (4)$$

$$\varpi_1^*(r,n)=\int_0^{h_1}\varpi_1(r,z)\cos\frac{n\pi z}{h_1}dz,\quad \varpi_1^2=E_3^{(1)}/G^{(1)}\quad . \qquad (5)$$

After applying the Weber transforms with respect to the coordinate r to equations (4) taking into account the boundary conditions, we obtain

$$(2/\pi)\varpi_1^*(a,n)-\lambda^2W_1(\lambda,n)-\varpi_1^2(n\pi/h_1)W_1(\lambda,n)=-\left((-1)^n/G_1\right)F(\lambda),\quad F(\lambda)=\int_a^\infty f(r)r\phi\lambda(r)dr,$$

$$W_1(\lambda,n)=\int_a^\infty\varpi_1^*(r,n)r\phi\lambda(r)dr,\quad \phi_\lambda(r)=I_0(a\lambda)Y_0(\lambda r)-Y_0(a\lambda)I_0(\lambda r)$$

I_ν, Y_ν - Bessel functions of the first and second kind.

After applying the inverse Weber transform and calculating some integrals, the need for the inverse Fourier transform remains. Dirichlet's theorem is used for this [6]. As a result, we get

$$\varpi_1(a,z)=\frac{2h_1}{\pi^2}\sum_{n=1}^\infty\left(\frac{f(a)}{E_3^{(1)}}(-1)^n-\frac{P_0^{(1)}}{E_cF_c}+\frac{2\pi}{E_cF_c}(-1)^nA_1(n)\right)\frac{\cos(n\pi z/h_1)}{n(n+g_1M_1(n))}$$

$$\left.\frac{d\varpi_1}{dr}\right|_{r=a} = \frac{2}{G^{(1)}ah_1}\sum_{n=1}^{\infty}(-1)^n A_1(n)\cos(n\pi z/h_1) -$$

$$-\frac{2\varpi_1}{\pi}\sum_{n=1}^{\infty}\left(\frac{f(a)}{E_3^{(1)}}(-1)^n - \frac{P_0^{(1)}}{E_cF_c} + \frac{2\pi}{E_cF_c}(-1)^n A_1(n)\right)\frac{\cos(n\pi z/h_1)}{nM_1^{-1}(n)+g_1}$$

$$f(r) = \int_0^{\infty}\left(F(\lambda)\phi_\lambda(r)\lambda d\lambda\right)/\left(J_0^2(a\lambda)+Y_0^2(a\lambda)\right)$$

For the second layer, the solution is carried out similarly, taking into account the change of variable $z - h_1 = \xi$ in the equations(3), (4).

Main results and their discussions

After estimating a number of integrals using the mean value theorem and taking into account the boundary conditions, we find the sought functions and their derivatives for two layers

$$w_1(a,z) = \frac{2h_1}{\pi^2}\frac{f(a)}{E_3^{(1)}}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi z/h_1)}{n(n+g_1M_1(n))} - \frac{2h_1}{\pi^2}\frac{P_0^{(1)}}{E_cF_c}\sum_{n=1}^{\infty}\frac{\cos(n\pi z/h_1)}{n(n+g_1M_1(n))}$$

$$w_2(a,\xi) = \frac{2h_2}{\pi^2}\frac{P_0^{(2)}}{E_cF_c}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi\xi/h_2)}{n(n+g_2M_2(n))} - \frac{2h_2}{\pi^2}\frac{f(a)}{E_3^{(2)}}\sum_{n=1}^{\infty}\frac{\cos(n\pi\xi/h_2)}{n(n+g_2M_2(n))}$$

$$\frac{dw_1(a,z)}{dz} = -\frac{2}{\pi}\frac{f(a)}{E_3^{(1)}}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi z/h_1)}{(n+g_1M_1(n))} - \frac{2}{\pi}\frac{P_0^{(1)}}{E_cF_c}\sum_{n=1}^{\infty}\frac{\sin(n\pi z/h_1)}{(n+g_1M_1(n))}$$

$$\frac{dw_2(a,\xi)}{dz} = -\frac{2}{\pi}\frac{P_0^{(2)}}{E_cF_c}\sum_{n=1}^{\infty}\frac{(-1)^n\sin(n\pi\xi/h_2)}{(n+g_2M_2(n))} + \frac{2}{\pi}\frac{f(a)}{E_3^{(2)}}\sum_{n=1}^{\infty}\frac{\sin(n\pi\xi/h_2)}{(n+g_2M_2(n))}$$

$$\left.\frac{\partial w_1}{\partial r}\right|_{r=a} = -\frac{2\omega_1}{\pi}\frac{f(a)}{E_3^{(1)}}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi z/h_1)}{nM_1^{-1}(n)+g_1} + \frac{2\omega_1}{\pi}\frac{P_0^{(1)}}{E_cF_c}\sum_{n=1}^{\infty}\frac{\cos(n\pi z/h_1)}{nM_1^{-1}(n)+g_1}$$

$$\left.\frac{\partial w_2}{\partial r}\right|_{r=a} = -\frac{2\omega_2}{\pi}\frac{P_0^{(2)}}{E_cF_c}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi\xi/h_2)}{nM_2^{-1}(n)+g_2} + \frac{2\omega_2}{\pi}\frac{f(a)}{E_3^{(2)}}\sum_{n=1}^{\infty}\frac{\cos(n\pi\xi/h_2)}{nM_2^{-1}(n)+g_2}$$

Unknown value $f(a)$ is found from condition $w_1(a_1, h_1) = w_2(a, 0)$.

$$f(a) = \frac{P_0^{(1)}E_3^1}{E_cF_c}\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+g_1M_1(n))} + \frac{h_2}{h_1}\frac{P_0^{(2)}}{P_0^{(1)}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+g_2M_2(n))}\right)/$$

$$\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+g_1M_1(n))} + \frac{h_2}{h_1}\frac{E_3^{(1)}}{E_3^{(2)}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n(n+g_2M_2(n))}\right).$$

The load in the stringer $N(z)$ and the load of contact interaction are according to the formulas $N(z) = E_cF_c dw(a,z)/dz, \qquad q(z) = -2\pi aG(\partial\varpi/\partial r)|_{r=a}$

If $h_2 = 0$ then we come to the problem of transferring the load from the bar to the thickness h_1 layer. If $h_2 = 0,\ h_1 \to \infty$ then we obtain a solution to the problem of transferring a load to an orthotropic semi-infinite viscoelastic body. Such passage to the limit allows you to check the results obtained by the proposed method, since the simplified versions of the problem have already been solved.

The inverse Laplace transform determines N and q depending on the coordinates and time. To go to the originals, the effort is presented in the form of rows in a small parameter $\bar{\varepsilon}$ which depends on p

$$N(z,p)=\left(N_0(z)+N_1(z)\bar{\varepsilon}+N_2(z)\bar{\varepsilon}^2+\ldots\right)/\,p$$
$$q(z,p)=\left(q_0(z)+q_1(z)\bar{\varepsilon}+q_2(z)\bar{\varepsilon}^2+\ldots\right)/\,p \qquad (6)$$

The coefficients of these expansions are easier to find for "small" and "large" values of the parameter. Such limiting values of functions can be connected using the Padé approximant [5].

For example, in the problem of transferring a load to a half-space, which has predominantly shear creep at $p\to\infty$, $\bar{\varepsilon}=k/(p+\beta)$, for $p\to 0, \bar{\varepsilon}=\Delta\, p/(p+\beta)$, $\Delta=-k/(\beta+k)$.

The originals of the functions for small values of time have the form

$$N(z,t)=N_0(z)+N_{10}(z)(K/\beta)(1-e^{-\beta t})+\ldots,$$

$$N_0(z)=\frac{2P_0}{\pi}\int_0^{\infty}\frac{\sin zs}{s+g_0M_0(s)}ds\,,$$

$$N_{10}(z)=\frac{2P_0}{\pi}\int_0^{\infty}\frac{\left[g_0a\omega_0 s\left(1-M_0^2(s)\right)+2g_0M_0(s)\right]\sin zs}{2\left[s+g_0M_0(s)\right]^2}ds,$$

$$q_0(z)=\frac{2P_0g_0}{\pi}\int_0^{\infty}\frac{\cos zs}{g_0+sM_0^{-1}(s)}ds, \qquad (7)$$

$$q_{10}(z)=\frac{2P_0g_0}{\pi}\int_0^{\infty}\frac{\left[a\omega_0 s^2\left(1-M_0^{-2}(s)\right)-2sM_0^{-1}(s)\right]\cos zs}{2\left[g_0+sM_0^{-1}(s)\right]^2}ds,$$

$$M_0(s)=K_1(a\omega_0 s)/K_0(a\omega_0 s).$$

At large times

$$N(z,t)=N_\infty(z)+N_{1\infty}(z)\Delta e^{-\chi t}+\ldots$$
$$q(z,t)=q_\infty(z)+q_{1\infty}(z)\Delta e^{-\chi t}+\ldots$$

The expansion coefficients are found from (7) after replacing ω_0 by ω_∞, g_0 by g_∞ for

$$\omega_\infty=\omega_0(1+k/\beta)^{1/2},\quad g_\infty=g_0(1+k/\beta)^{-1/2}.$$

For the same case, one of the Padé approximants has the form

$$N^*=\frac{0.915-0.11t-1.31e^{0.5t}}{1-0.134t-1.55e^{0.5t}}.$$

Figures 1, 2 show the limiting values $N^* = N / P_0$ and $q^* = q / P_0 g_0$ at $t \to 0$ (curve 1) and $t \to \infty$ (curve 2).

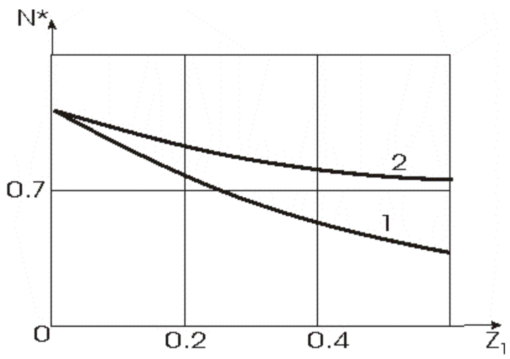

Figure 1. The core tension change

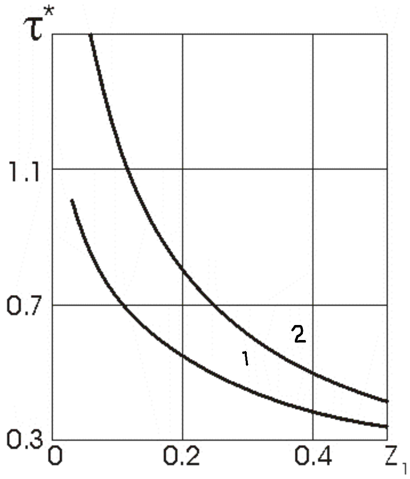

Figure 2. Altering contact efforts

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Conclusions

The practical significance of the results obtained lies in the possibility of using the proposed approach to construct reasonable simplified boundary value problems that bring the model as close as possible to the real problem. The results can be used for numerical implementation in the design of complex multi-layer structures from modern materials. The obtained solutions of the spatial contact problem can be used as exact solutions in the estimation of approximate computations.

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