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ANALYSIS OF THE INTERACTION OF INERTIAL FORCES IN A STABILITY PROBLEM

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Using the plane of characteristic indices we study the stability of an elastic filament with a distributed inertial load moving along it at constant speed. Taking account of the Coriolis inertial forces and assuming the presence of dissipative forces, we carry out a qualitative analysis of the results for different relations among the parameters of the system; this analysis is in agreement with the general stability theorems for elastic structures.

As is known, the validity of an approach to solving the problem of stability of an elastic structure is determined by the nature of the system being studied. The classical engineering approach in such problems is based on the assumption that the model in question is a nongyroscopic conservative system. Consequently, the lack of agreement as to stability or instability may arise from imprecise assumptions on the nature of the forces, as well as their interaction in the given model. Thus, if the system is not purely nongyroscopic, simple static approaches are usually not valid [1]. Moreover, even the dynamic method becomes dubious if physical nonidealities are not taken into account. The uncertainty in the initial assumptions can be compensated for by analyzing a specific simple system, which makes it possible to pass to certain generalizations applicable to more complicated systems, and caution is observed by the choice of this method of studying them.

The object of study is an elementary mechanical system—an elastic filament with a distributed inertial load moving along it at a constant speed. Many papers devoted to the construction of the solutions of the differential equations that describe the motion of such systems and to the study of their stability are known. In view of certain complications in the application of the method of separation of variables to the equations of motion, and in order to simplify them, one sometimes resorts to different assumptions. The most typical is the assumption that the critical speed of motion of the load is much faster than real speeds; this assumption gives a basis for the researcher to simplify the equation by neglecting terms with mixed partial derivatives, one of which corresponds to the inertial Coriolis forces. Although there are grounds for such an assumption in many cases, so that we need not enter into a discussion of the validity of this approach, nevertheless the qualitative side of the nature of the interaction of the various inertial forces in mechanical systems of such a type is lost from view.

We shall study the qualitative side of the general topic resulting from the influence of different forces (external and internal) on the nature of the oscillations of elementary mechanical objects belonging to the family of systems with a moving inertial load.

The mathematical model of these elastic systems is the so-called generalized vibrating string equation [3] (Table 1). The mathematical model that takes account of the rotational inertia of sections was studied in [2]. All equations in the table can be written in the following dimensionless unified form:

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma_2 \frac{\partial^2 u}{\partial x \partial t} + \gamma_1 \frac{\partial^2 u}{\partial x^2} = 0. \tag{1}$$

The values of the dimensionless coefficients for all types of equations of the form (1) are given in Table 2.

The dynamic individuality of a linear vibrating system is characterized most completely by the characteristic frequencies and characteristic shapes of its vibrations. On the other hand, in practical applications it is no less important to establish which relation among the parameters of the mechanical system leads to a transition from stability to instability. Euler's method, which gives the correct solution in a number of cases, is universal in comparison with the more general dynamic method based on the vibrations of the system about a quasistatic equilibrium position. In this connection we limit ourselves to the study of the

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Table 1

Mechanical string model	Mathematical model $c^2 = \dfrac{T}{\rho}$; $\varepsilon = \dfrac{\rho'}{\rho + \rho'}$	ε	V
Classical	$\dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$	0	0
Fixed load	$\dfrac{\partial^2 u}{\partial t^2} + (\varepsilon - 1) c^2 \dfrac{\partial^2 u}{\partial x^2} = 0$	$0 < \varepsilon < 1$	0
With moving load	$\dfrac{\partial^2 u}{\partial t^2} + 2\varepsilon V \dfrac{\partial^2 u}{\partial x \partial t} + (\varepsilon V^2 + (\varepsilon - 1)c^2) \dfrac{\partial^2 u}{\partial x^2} = 0$	$0 < \varepsilon < 1$	$\neq 0$
Moving with load	$\dfrac{\partial^2 u}{\partial t^2} + 2V \dfrac{\partial^2 u}{\partial x \partial t} + (V^2 + (\varepsilon - 1)c^2) \dfrac{\partial^2 u}{\partial x^2} = 0$	$0 < \varepsilon < 1$	$\neq 0$
Moving without load	$\dfrac{\partial^2 u}{\partial t^2} + 2V \dfrac{\partial^2 u}{\partial x \partial t} + (V^2 - c^2) \dfrac{\partial^2 u}{\partial x^2} = 0$	0	$\neq 0$

Table 2

Model	Equation types 1	2	3	4	5
α_2	0	0	$\varepsilon V/c$	V/c	V/c
α_1	-1	$\varepsilon - 1$	$\dfrac{\varepsilon V^2}{c^2} + \varepsilon - 1$	$\dfrac{V^2}{c^2} + \varepsilon - 1$	$\dfrac{V^2}{c^2} - 1$

character of small vibrations, following [1]. Instead of the vibration frequencies we introduce the characteristic indices s connected with the frequencies by the simple relation $s = i\omega$, where ω is the complex frequency. Equation (1) can be satisfied by a function of the form

$$u(x_0, t) = X(x) e^{st}, \tag{2}$$

where $X(x)$ is a function that characterizes the shape of the vibrations. Substitution of the expression (2) into Eq. (1) leads to an ordinary differential equation

$$\gamma_1 X'' + 2s\gamma_2 X' + s^2 X = 0. \tag{3}$$

We write the integral of Eq. (3) as

$$X(x) = C_1 e^{k_1 x} + C_2 e^{k_2 x}, \tag{4}$$

where k_1 and k_2 are the roots of the characteristic equation analogous to the differential equation (3), and the solution (4) must satisfy the zero boundary conditions $X(0) = 0$, $X(1) = 0$, corresponding to a string with its ends clamped. The condition that the solution of Eq. (3) be nontrivial leads to the relation $k_2 - k_1 = 2n\pi i$, whence

$$s_n = \pm i\left(n\pi\gamma_1 / \sqrt{\gamma_2^2 - \gamma_1}\right). \tag{5}$$

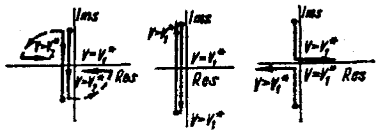
We shall study the behavior of the characteristic indices s_n under variation of the parameter V (the speed of the inertial load) that occurs in the coefficients γ_2 and γ_1.

As can be seen from (5), all the characteristic indices, which remain purely imaginary as the parameter V increases, must vanish when $\gamma_1 = 0$, then change sign and, increasing in absolute value, become real when the condition $\gamma_2^2 - \gamma_1 < 0$ holds.

Thus we obtain the critical values of the parameter

$$V_1^{*2} = ((1/\varepsilon) - 1)c^2; \tag{6}$$

$$V_2^{*2} = c^2/\varepsilon; \tag{7}$$



(a) (b) (c)

Table 3

Model	Equation types		
	3	4	5
V_1^{*2}	$\left(\frac{1}{\varepsilon}-1\right)c^2$	$(1-\varepsilon)c^2$	c^2
V_2^{*2}	c^2/ε	–	–

Remark. The instability for all types of equations is static: the region of uncertainty for (3) is $V_1^* < V < V_2^*$; for (4) it is $V > V_1^*$; for (5) it is $V > V_1^*$.

and the remaining parameters of the system under consideration are assumed constant. The critical values of the speed V for all types of equations of the form (1) are given in Table 3.

The behavior of the characteristic indices is shown in the figure. In the case of a string with a moving inertial load (type 3, Fig. *a*) the vanishing of the characteristic indices corresponds to "neutral" equilibrium, i.e., the presence of other nearby equilibrium shapes (in addition to the original equilibrium shape). At load speeds larger than V_1^* the characteristic indices, after passing through the origin into the lower half-plane, increase in absolute value and become infinite at V_2^*. This value of the speed also corresponds to the passage of the indices into the right half-plane.

Thus the speed value V_1^* is critical in the sense of Euler. At a load speed larger than V_2^* the motion of the string will become an aperiodic recession from the equilibrium state. Consequently the given system is characterized by static instability with passage through the point at infinity [1].

The behavior of the characteristic indices with a moving string (types 4 and 5) is shown in Fig. *b*. For any value of the speed they remain on the imaginary axis. Passage through the origin at the speed V_1^* corresponds to loss of stability in the form of a branching of equilibrium shapes. The perturbed motion is oscillatory with constant amplitudes depending on the initial conditions. The behavior of the characteristic indices in the case $\gamma_2 = 0$, i.e., ignoring the Coriolis inertial force, is the same for types 3, 4, and 5 (Fig. *c*). The study of the simplified equation leads to the same result in all cases using either the static or the dynamic approach. Loss of stability occurs as a static instability. All the characteristic indices leave the imaginary axis at the speed V_1^*, i.e., the perturbed motion is aperiodic for all $V > V_1^*$.

The interaction of the Coriolis inertial forces with external resistance and their mutual influence on the stability of motion of the system under consideration is of interest. In studying this question we shall use the classical theory of the influence of external resistance on free and forced vibrations, which is based on the assumption that the resisting forces are linear functions of the speed of displacement of elements of the filament.

When we take account of external damping, Eq. 1 assumes the form

$$\frac{\partial^2 u}{\partial t^2} + 2\gamma_3\frac{\partial u}{\partial t} + 2\gamma\frac{\partial^2 u}{\partial x \partial t} + \gamma_1\frac{\partial^2 u}{\partial x^2} = 0; \tag{8}$$

here $\gamma_3 = hl/(c(\rho + \rho^1))$, where h is the damping coefficient relative to the total mass of elements of the filament and the moving load. Assuming a solution of Eq. (8) in the form (2) and repeating the computations given above for the characteristic indices, we obtain the expression

$$s_n = \frac{\gamma_1\gamma_2}{\gamma_2^2\gamma_1} \pm i\sqrt{\frac{(n\pi)^2\gamma_1^2}{\gamma_2^2-\gamma_1} - \left(\frac{\gamma_1\gamma_2}{\gamma_2^2-\gamma_1}\right)^2}, \tag{9}$$

which makes it possible to trace the behavior of the characteristic indices as the parameter V is varied. The characteristic indices (9) form complex-conjugate pairs which, when they are in the open half-plane of negative real values, move toward the origin as the speed of the load increases. At $\gamma_1 = 0$ each pair of indices coalesces at the origin and moves away from it into the open half-plane of positive real values, moving toward the real axis. Thus the real part of the indices remains negative for values of the parameter $V < V_1^*$. At a certain value of the parameter V

$$V^2 = V_2^{*2} - \alpha_n, \tag{10}$$

where α_n is determined from the condition of positivity of the expression under the radical in (9), some of them reach the positive real axis, and remain on it for higher values of the speed. In this case the transition of the characteristic indices to the right half-plane occurs through the origin at the parameter value V_1^*, i.e., the system in question is characterized by static instability. It is interesting that, at least for values of the parameter V near V_1^*, but larger than it, the motion of the string is oscillatory with increasing amplitudes. For values of the parameter V larger than the value defined by (10) the oscillatory motion is replaced by an aperiodic motion. For systems of type 4, 5 the transition of the indices to the right half-plane also occurs through the origin at the parameter value V_1^*. The subsequent motion is oscillatory with amplitudes increasing in time.

This analysis shows that external damping decreases the stabilizing role of Coriolis inertial forces. For that reason, when studying the stability of such a system, it does not make sense to take account of both of these forces. Nevertheless, taking account of the forces just mentioned and others gives some idea of the post-critical behavior and makes it possible to predict the dynamic loss of stability in the case of moving objects of finite length (types 4 and 5). Since the static instability in these systems at $V = V_1^*$ is independent of the dissipative and gyroscopic forces, the systems can be compared with the nongyroscopic conservative system obtained from them by ignoring these forces. Moreover, the stability of the system is very sensitive to all kinds of seemingly unimportant effects. This circumstance must be taken into account in interpreting the results obtained using approximate methods of solution or in the analysis of simplified models. The conclusions of this study are in agreement with the general theorems on stability of elastic systems [4].

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